Suppose $f(y|\theta)$ is the general form of the density, that is
$f(y|\theta) \in$ exponential family of distributions.

Then

$$f(y|\theta) = s(y)\,t(\theta)\,e^{a(y)b(\theta)}$$

OR $$f(y|\theta) = \exp\left(a(y)b(\theta) + c(\theta) + d(y)\right)$$

Where $s(y) = \exp(d(y))$; $t(\theta) = \exp(c(\theta))$

N.B. if $a(y) = y$ then 'we say' the distribution is in canonical form!

$$\boxed{\begin{array}{c} a(y) = y \\ \text{CANONICAL} \end{array}}$$

$b(\theta)$ is called the natural parameter.

θ is the parameter of interest, all other parameters are regarded as constants or nuisance parameters.

Example: Poisson distribution

$$f(y|\theta) = \frac{\theta^y e^{-\theta}}{y!} = \exp\left[y\log\theta - \theta - \log y!\right]$$

compare with $\exp\left(a(y)b(\theta) + c(\theta) + d(y)\right)$

$a(y) = y$, $b(\theta) = \log(\theta)$ ie. 1. Canonical
2. nat. parameter $= \log(\theta)$.

$\int_Y f(y|\theta)\,dy = 1$ ppty of any density.

$$\frac{d}{d\theta}\int f(y|\theta)\,dy = \frac{d}{d\theta}1 = 0$$

move derivative inside integral

$$\int \frac{df(y|\theta)}{d\theta}\,dy = 0 \quad \text{and } \underline{\text{again}} \quad \int \frac{d^2 f(y|\theta)}{d\theta^2}\,dy = 0$$

But since $f(y|\theta) \in$ exponential family

$$f(y|\theta) = \exp\left(a(y)\,b(\theta) + c(\theta) + d(y)\right)$$

$$\frac{df}{d\theta} = \left[a(y)\,b'(\theta) + c'(\theta)\right] f(y|\theta)$$

$$\therefore \int \left[a(y)\,b'(\theta) + c'(\theta)\right] f(y|\theta)\,dy = 0$$

$$\int b'(\theta)\,a(y)\,f(y|\theta)\,dy + \int c'(\theta)\,f(y|\theta)\,dy = 0$$

$$b'(\theta)\int a(y)\,f(y|\theta)\,dy + c'(\theta)\int f(y|\theta)\,dy = 0$$

$$b'(\theta)\,E(a(y)) + c'(\theta) = 0$$

$$\boxed{\therefore\ E(a(y)) = \frac{-c'(\theta)}{b'(\theta)}}$$

$$\frac{d^2 f(y|\theta)}{d\theta^2} = [a(y)b''(\theta) + c''(\theta)]\, f(y|\theta) \;+$$
$$[a(y)b'(\theta) + c'(\theta)]^2 f(y|\theta)$$

The second term can be simplified!

$$c'(\theta) = -b'(\theta)E(a(y)) \quad \text{— substitute into}$$

$$[a(y)\, b'(\theta) - b'(\theta)E(a(y))]^2$$

$$= b'(\theta)^2 \left[a(y) - E(a(y))\right]^2$$

$$\therefore \int \frac{d^2 f(y|\theta)}{d\theta^2}\, dy = \int (b''(\theta)\, a(y) + c''(\theta))\, f(y|\theta)\, dy$$

$$+\, b'(\theta)^2 \operatorname{var}(a(Y))$$

$$= b''(\theta)E(a(Y)) + c''(\theta) + b'(\theta)^2 v(a(y)) = 0$$

$$\Rightarrow \operatorname{var}(a(Y)) = \frac{-b''(\theta)E(a(Y)) - c''(\theta)}{b'(\theta)^2}$$

$$\text{but } E(a(Y)) = \frac{-c'(\theta)}{b'(\theta)}$$

$$\boxed{\therefore \operatorname{var}(a(Y)) = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{b'(\theta)^3}}$$

$$E(a(Y)) = \frac{-c'(\theta)}{b'(\theta)}$$

for Poisson $a(Y) = Y$, $b(\theta) = \log(\theta)$, $c(\theta) = -\theta$, $d(y) = -\log y!$

$$E(a(Y)) = E(Y) = \frac{-(-1)}{\frac{1}{\theta}} = \theta$$

$$V(a(Y)) = \frac{b''(\theta)\,c'(\theta) - c''(\theta)\,b'(\theta)}{b'(\theta)^3}$$

$$= \frac{-\frac{1}{\theta^2}(-1) - 0}{\left(\frac{1}{\theta}\right)^3}$$

$$= \theta$$

For a Poisson $E(Y) = \theta$ & $V(Y) = \theta$